Notes: Equiangular Tight Frames via Geometric Invariant Theory

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Chapter 1

Preliminaries

1.1 Motivation:

Let V be a K vector space where K is algebraically closed.

Definition 1. *Let* $Fr_n(V) := \{(v_i)_{i=1}^n \in V^n \mid v_i \neq \lambda v_j,$ *for* $i \neq j$, *span*$(\{v_i\}) = V\}$ *be called set of frames.*

We define some actions of different groups on $Fr_n(V) \times Fr_n(V^*)$ to get a notion of equivalent frames:

- $GL(V)$ acts on the left as : $M((v_i), (\phi_i)) = ((Mv_i), (v \mapsto \phi_i(M^{-1}v)))$
- $(K^\times)^n \rtimes S_n$ acts on the right as (scaling and permuting frames):
  - $((v_i), (\phi_i))\lambda_i = ((\lambda_i v_i), (v \mapsto \phi_i(\lambda_i^{-1} v)))$
  - $((v_i), (\phi_i))\sigma = ((v_{\sigma(i)})(\phi_{\sigma(i)}))$

Definition 2. *Now we need to look at the subspace of* $GL(V)\backslash Fr_n(V) \times Fr_n(V^*)/(K^\times)^n \rtimes S_n$ *called* $\boldsymbol{ETB_n(V)}$ *which satisfies the following:*

1. $\phi_i(v_i) = 1$
2. $\phi_i(v_j)\phi_j(v_i) = \frac{n-d}{d(n-1)}$ *for* $i \neq j$
3. $v_j = \frac{d}{n} \sum_{i=1}^{n} \phi_i(v_j) v_i$

Now the **BIG QUESTIONS** are:

- what is $ETB_n(V)$? Variety?
- when is dimension of $ETB_n(V)$ zero? In that case, what is the number of points $\#ETB_n(V)$?

Special Case: Equiangular Tight Frames

$K = \mathbb{C}$ complex numbers, $\phi_i = v_i^\dagger \implies \phi_i(v) = \langle v_i, v \rangle = v_i^\dagger v$ is called **ETF**.

1. unit norm: $\langle v_i, v_i \rangle = 1$
2. equiangular: $| \langle v_i, v_j \rangle |^2 = \dfrac{n-d}{d(n-1)}$
3. tight: $v = \frac{d}{n} \sum_{i=1}^{n} \langle v_i, v \rangle v_i$

1.2 Motivation for GIT